

# Embedded constant $m^{\text{th}}$ mean curvature hypersurfaces on spheres \*

Guoxin Wei and Guohua Wen

## Abstract

In this paper, we study  $n$ -dimensional hypersurfaces with constant  $m^{\text{th}}$  mean curvature  $H_m$  in a unit sphere  $S^{n+1}(1)$  and prove that if the  $m^{\text{th}}$  mean curvature  $H_m$  takes value between  $\frac{1}{(\tan \frac{\pi}{k})^m}$  and  $\frac{k^2-2}{n}(\frac{k^2+m-2}{n-m})^{\frac{m-2}{2}}$  for  $1 \leq m \leq n-1$  and any integer  $k \geq 2$ , then there exists at least one  $n$ -dimensional compact nontrivial embedded hypersurface with constant  $H_m > 0$  in  $S^{n+1}(1)$ . When  $m = 1$ , our results reduce to the results of Perdomo [12]; when  $m = 2$  and  $m = 4$ , our results reduce to the results of Cheng-Li-Wei [14].

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## 1. Introduction

It is well known that Alexandrov [1] and Montiel-Ros [10] proved that the standard round spheres are the only possible oriented compact embedded hypersurfaces with constant  $m^{\text{th}}$  mean curvature  $H_m$  in a Euclidean space  $\mathbb{R}^{n+1}$ , for  $m \geq 1$ . For hypersurfaces in a unit sphere  $S^{n+1}(1)$ , standard round spheres and Clifford hypersurfaces  $S^l(a) \times S^{n-l}(b)$ ,  $1 \leq l \leq n-1$  are compact embedded hypersurfaces in  $S^{n+1}(1)$ . Hence the following problem is interesting (also see [2], [6], [14]):

**Problem:** Do there exist compact embedded hypersurfaces with constant  $m^{\text{th}}$  mean curvature  $H_m$  in  $S^{n+1}(1)$  other than the standard round spheres and Clifford hypersurfaces?

When  $m = 1$ , namely, when the mean curvature is constant, Ripoll [13] has proved the existence of compact embedded hypersurfaces of  $S^3(1)$  with constant mean curvature ( $H \neq 0, \pm \frac{\sqrt{3}}{3}$ ) other than the standard round spheres and the Clifford hypersurfaces. For general  $n$ , Perdomo [12] has proved

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**Theorem 1.1** (Main Theorem of [12]). *For any  $n \geq 2$  and any integer  $k \geq 2$ , if mean curvature  $H$  takes value between  $\frac{1}{(\tan \frac{\pi}{k})}$  and  $\frac{(k^2 - 2)\sqrt{n-1}}{n\sqrt{k^2 - 1}}$ , then there exists an  $n$ -dimensional compact nontrivial embedded hypersurface with constant mean curvature  $H > 0$  in  $S^{n+1}(1)$ .*

For  $m = 2$ , that is, when the scalar curvature is constant, Cheng, Li and Wei [14] has proved

**Theorem 1.2** ([14]). *For any  $n \geq 3$  and any integer  $k \geq 2$ , if  $H_2 = \frac{R-n(n-1)}{n(n-1)}$  takes value between  $\frac{1}{(\tan \frac{\pi}{k})^2}$  and  $\frac{k^2 - 2}{n}$ , then there exists an  $n$ -dimensional compact nontrivial embedded hypersurface  $M$  with constant 2-th mean curvature  $H_2 > 0$  (i.e. scalar curvature  $R > n(n-1)$ ) in  $S^{n+1}(1)$ , where  $R$  is the scalar curvature of  $M$ .*

For  $m = 4$ , Cheng, Li and Wei [14] has proved

**Theorem 1.3** ([14]). *For any  $n \geq 5$  and any integer  $k \geq 3$ , if 4<sup>th</sup> mean curvature  $H_4$  takes value between  $\frac{1}{(\tan \frac{\pi}{k})^4}$  and  $\frac{k^4 - 4}{n(n-4)}$ , then there exists an  $n$ -dimensional compact nontrivial embedded hypersurface with constant  $H_4 > 0$  in  $S^{n+1}(1)$ .*

For general  $1 \leq m \leq n-1$ , we will prove that there exist many compact nontrivial embedded hypersurfaces with constant  $m^{\text{th}}$  mean curvature  $H_m > 0$  in  $S^{n+1}(1)$ , for  $1 \leq m \leq n-1$ . In fact, we prove

**Theorem 1.4.** *For  $1 \leq m \leq n-1$  and any integer  $k \geq 2$ , if  $m^{\text{th}}$  mean curvature  $H_m$  takes value between  $\frac{1}{(\tan \frac{\pi}{k})^m}$  and  $\frac{k^2 - 2}{n} \left( \frac{k^2 + m - 2}{n - m} \right)^{\frac{m-2}{2}}$ , then there exists at least one  $n$ -dimensional compact nontrivial embedded hypersurface with constant  $H_m > 0$  in  $S^{n+1}(1)$ .*

**Remark 1.1.** *For  $m = 1$ , Theorem 1.4 reduces to the conclusion of Perdomo [12]. For  $m = 2, m = 4$ , Theorem 1.4 reduces to the results of Cheng-Li-Wei [14].*

## 2. Proof of Theorem

Using the same notations as those of [14], we can have

$$(g')^2 = q(g), \quad \text{where } q(v) = C - v^2(v^{-n} + H_m)^{\frac{2}{m}} - v^2, \quad (2.1)$$

$$\begin{aligned} & q'(v) \\ &= 2v \left\{ -(v^{-n} + H_m)^{\frac{2}{m}} + \frac{n}{m} v^{-n} (v^{-n} + H_m)^{\frac{2-m}{m}} - 1 \right\} \\ &= -2v \left\{ (v^{-n} + H_m)^{\frac{2-m}{m}} \left[ \frac{m-n}{m} v^{-n} + H_m \right] + 1 \right\}, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned}
& q''(v) \\
&= -\frac{2(v^{-n} + H_m)^{\frac{2-2m}{m}}}{m^2} \left\{ (2n^2 - 3nm + m^2)v^{-2n} + m(n^2 - 3n + 2m)H_mv^{-n} + m^2H_m^2 \right\} \\
&\quad - 2 \\
&< -2.
\end{aligned} \tag{2.3}$$

From (2.3), one obtains that  $q'(v)$  is a decreasing function of  $v$  in  $[0, +\infty)$ . From (2.2), one has  $q'(v) > 0$  if  $v \rightarrow 0$ ;  $q'(v) < 0$  if  $v \rightarrow \infty$ . Hence there exists  $0 < v_0 < \infty$  such that  $q'(v_0) = 0$ . Moreover, the function  $q(v)$  is a monotone increasing function of  $v$  in  $(0, v_0]$  and decreasing function of  $v$  in  $[v_0, +\infty)$ . Hence, for some value of  $C$ , the function  $q$  has two positive roots  $t_1$  and  $t_2$ , such that  $t_1 \leq t_2$ ,  $q(t_1) = q(t_2) = 0$  and  $q(t) > 0$  if  $t \in (t_1, t_2)$ .

From the results of section 4 of [14], we have

$$P(H_m, n, C) = 2 \int_0^{\frac{T}{2}} \frac{r(s)\lambda(s)}{1 - r^2(s)} ds. \tag{2.4}$$

Since  $r(s) = \frac{g(s)}{\sqrt{C}}$  and  $\lambda(s) = (g(s)^{-n} + H_m)^{\frac{1}{m}}$ , then it follows from (2.4) that

$$P(H_m, n, C) = 2 \int_0^{\frac{T}{2}} \frac{\sqrt{C}g(s)(g(s)^{-n} + H_m)^{\frac{1}{m}}}{C - g^2(s)} ds. \tag{2.5}$$

Doing the substitution  $t = g(s)$ , putting  $c_0 = C - q(v_0)$ ,  $-2a = q''(v_0)$  and applying Lemma 5.1 of [14], one concludes from  $g(0) = t_1$  and  $g(\frac{T}{2}) = t_2$  that

$$\begin{aligned}
& \lim_{C \rightarrow c_0^+} P(H_m, n, C) \\
&= \frac{2\pi\sqrt{c_0}}{\sqrt{a}\sqrt{c_0 - v_0^2}} \\
&= \frac{2m\pi \left( (v_0^{-n} + H_m)^{\frac{2m-2}{m}} + (v_0^{-n} + H_m)^{\frac{2m-4}{m}} \right)^{\frac{1}{2}}}{\left( (2n^2 - 3mn + m^2)v_0^{-2n} + m(n^2 - 3n + 2m)H_mv_0^{-n} + m^2H_m^2 + m^2(v_0^{-n} + H_m)^{\frac{2m-2}{m}} \right)^{\frac{1}{2}}}.
\end{aligned} \tag{2.6}$$

Writing

$$F_0 = (v_0^{-n} + H_m)^{\frac{1}{m}}, \tag{2.7}$$

then  $F_0 > 0$  and  $q'(v_0) = 0$  can be written as

$$F_0^m + \frac{m}{m-n}F_0^{m-2} + \frac{n}{m-n}H_m = 0. \tag{2.8}$$

We next compute the the numerator and denominator of  $P(H_m, n, C)$  by using of (2.8). By a direct calculation, we know

$$\begin{aligned}
& 2m\pi \left( (v_0^{-n} + H_m)^{\frac{2m-2}{m}} + (v_0^{-n} + H_m)^{\frac{2m-4}{m}} \right)^{\frac{1}{2}} \\
&= 2m\pi \left( F_0^{m-2} \left( \frac{m}{n-m} F_0^{m-2} + \frac{n}{n-m} H_m \right) + F_0^{2m-4} \right)^{\frac{1}{2}} \\
&= 2m\pi \left( \frac{n}{n-m} \right)^{\frac{1}{2}} F_0^{\frac{m-2}{2}} \left( F_0^{m-2} + H_m \right)^{\frac{1}{2}},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
& (2n^2 - 3mn + m^2)v_0^{-2n} + m(n^2 - 3n + 2m)H_mv_0^{-n} + m^2H_m^2 \\
&+ m^2(v_0^{-n} + H_m)^{\frac{2m-2}{m}} \\
&= (n-m)(2n-m) \left( v_0^{-n} + H_m \right)^2 + n(-4n + mn + 3m)(v_0^{-n} + H_m)H_m \\
&+ n^2(2-m)H_m^2 + m^2(v_0^{-n} + H_m)^{\frac{2m-2}{m}} \\
&= (n-m)(2n-m) \left( \frac{m}{n-m} F_0^{m-2} + \frac{n}{n-m} H_m \right)^2 \\
&+ n(-4n + mn + 3m) \left( \frac{m}{n-m} F_0^{m-2} + \frac{n}{n-m} H_m \right) H_m + n^2(2-m)H_m^2 \\
&+ m^2 F_0^{m-2} \left( \frac{m}{n-m} F_0^{m-2} + \frac{n}{n-m} H_m \right)^{\frac{2m-2}{m}} \\
&= \frac{m^2 n}{n-m} \left( F_0^{m-2} + H_m \right) \left( 2F_0^{m-2} + nH_m \right).
\end{aligned} \tag{2.10}$$

According (2.6), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}
& \lim_{C \rightarrow c_0^+} P(H_m, n, C) \\
&= \frac{2\pi\sqrt{c_0}}{\sqrt{a}\sqrt{c_0 - v_0^2}} \\
&= \frac{2m\pi \left( (v_0^{-n} + H_m)^{\frac{2m-2}{m}} + (v_0^{-n} + H_m)^{\frac{2m-4}{m}} \right)^{\frac{1}{2}}}{\left( (2n^2 - 3mn + m^2)v_0^{-2n} + m(n^2 - 3n + 2m)H_mv_0^{-n} + m^2H_m^2 + m^2(v_0^{-n} + H_m)^{\frac{2m-2}{m}} \right)^{\frac{1}{2}}} \\
&= 2\pi \left( \frac{F_0^{m-2}}{(n-m)F_0^m - (m-2)F_0^{m-2}} \right)^{\frac{1}{2}} \\
&= 2\pi \left( \frac{1}{(n-m)F_0^2 - (m-2)} \right)^{\frac{1}{2}},
\end{aligned} \tag{2.11}$$

and

$$F_0^2 > \frac{m-2}{n-m}. \quad (2.12)$$

On the other hand, we know that

$$\lim_{C \rightarrow \infty} P(H_m, n, C) = 2 \arctan \frac{1}{H_m^{\frac{1}{m}}}. \quad (2.13)$$

Therefore, for any fixed  $H_m > 0$ , the function  $P(H_m, n, C)$  takes all the values between

$$A(H_m) = 2 \arctan \frac{1}{H_m^{\frac{1}{m}}}, \quad B(H_m) = 2\pi \left( \frac{1}{(n-m)F_0^2 - (m-2)} \right)^{\frac{1}{2}}. \quad (2.14)$$

By a direct calculation, we obtain  $A(H_m)$  is a decreasing function of  $H_m$ . From (2.8) and (2.12), it is not hard to prove  $F_0$  is an increasing function of  $H_m$ , then  $\lim_{C \rightarrow c_0^+} P(H_m, n, C)$  is a decreasing function of  $H_m$ . Moreover, if  $2\pi \left( \frac{1}{(n-m)F_0^2 - (m-2)} \right)^{\frac{1}{2}} = \frac{2\pi}{k}$ , then

$$F_0^2 = \frac{k^2 + m - 2}{n - m}, \quad (2.15)$$

from (2.8), we obtain

$$H_m = \frac{k^2 - 2}{n} \left( \frac{k^2 + m - 2}{n - m} \right)^{\frac{m-2}{2}}, \quad (2.16)$$

thus,

$$A\left(\frac{1}{(\tan \frac{\pi}{k})^m}\right) = B\left(\frac{k^2 - 2}{n} \left( \frac{k^2 + m - 2}{n - m} \right)^{\frac{m-2}{2}}\right) = \frac{2\pi}{k}, \quad (2.17)$$

where  $k \geq 2$  is any integer, then we deduce that the number  $\frac{2\pi}{k}$  lies between  $A(H_m)$  and  $B(H_m)$ . Hence, by the continuity of  $P(H_m, n, C)$ , there exists some constant  $C_1$  such that  $P(H_m, n, C_1) = \frac{2\pi}{k}$ . If the period is  $\frac{2\pi}{k}$ , then there exists a compact embedded hypersurface with constant  $H_m$  which is not isometric to a round sphere or a Clifford hypersurface. We complete the proof of Theorem 1.4.

□

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Guoxin Wei  
 School of Mathematical Sciences  
 South China Normal University  
 510631, Guangzhou  
 China  
 E-mail: weigx@scnu.edu.cn  
 weigx03@mails.tsinghua.edu.cn

Guohua Wen  
 School of Mathematical Sciences  
 South China Normal University  
 510631, Guangzhou  
 China